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THE CREEP DEFORMATION OF VIBRATING STRUCTURES

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Abstract—A method is described of evaluating the creep deformation rate of a body subject to combined static and dynamic loading. The method is applied to a beam problem and it is shown that the solution may be expressed in terms of a reference stress history which possesses both a static and a dynamic component, independent of material parameters. Experiments on lead beams are shown to correlate quite closely with predicted behaviour.

INTRODUCTION

The analysis of the creep behaviour of a structure subjected to the variety of forces which occur during its design life remains amongst the more intractable problems of structural analysis. Inherent difficulties in such analyses arise in the description of the material behaviour as well as in the complexity of the necessarily non-linear structural analysis. Provided an adequate material description can be found, in principle it is possible to carry through a complete numerical solution, although, in practice the information gained does not always justify the considerable effort involved.

The situation is exemplified by the class of problems considered in this paper, which typifies the creep problems which occur in power producing plant. We assume that the structure is subjected to two distinct classes of loading. Constant loads exist which represent various types of dead loading and pressures. Superimposed upon these is a dynamic loading which causes vibration of the structure. Individually neither loading system would be likely to constitute a creep problem but we are concerned with the effect of their interaction through the non-linear material characteristics.

The purpose of this paper is to demonstrate that considerable insight into the problem may be gained through relatively simple calculations. The method described is based upon bounding theorems derived by the author[1], although the solution method may be understood without reference to these bounding solutions. The material behaviour is described by an elastic/viscous constitutive relationship, which, for uniaxial stress σ possesses the form

$$\dot{\epsilon} = \dot{e} + \dot{v}$$
$$\dot{e} = \dot{\sigma}/E \quad \text{and} \quad \dot{v}/\dot{v}_0 = (\sigma/\sigma_0)^n \quad (1)$$

where e and v denote elastic and viscous strains respectively. The constant n will be taken as an odd integer and creep rate $\dot{v}_0$ corresponds to stress σ_0. It cannot be claimed that these equations provide an adequate description of the behaviour of metals under rapid cyclic stresses, and it is used here as a model material which contains many of the elements of behaviour which are observed when stationary creep conditions are achieved. The analysis finally produces a "reference stress" history for the structure and loading history which is independent of the creep index n, and is therefore, in a restricted sense, independent of the details of the material description.

The analysis relies upon the following observation:- In a structure subjected to both static and dynamic loading two distinct time scales are involved. The most obvious time scale is that observed in the dynamic loading, for example the cycle time of the lowest resonance frequency, which for all practical structures may be considered as being measured in seconds or fractions of a second. The other time scale is provided by the material characteristics. A material time scale can be defined as the time taken for creep strain v to accumulate to the value of the elastic strain at a given stress. As the material model is non-linear this time will depend upon the stress and for a structure a suitable stress will be either the maximum or some mean stress. Let us suppose that we require a structure to survive for one year with a maximum accumulated creep strain of 1% say, which may be expected to be of the order of 10 to 20 elastic strains (these values are very approximate). Hence it is necessary to design the structure with stress levels such that the material time scale is therefore several orders of magnitude greater than the loading cycle. Of course, many practical design situations require design lives of up to 30 years and the material time scale is then increased even further.

The connection between this observation and bounding theory is described in a recent paper [1] for quasi-static loading. For cyclic loading it was shown that an upper bound on the total energy dissipated (by formation of creep strains) was bounded from above by the dissipation associated with the stress history

$$\underset{\sim}{\sigma}^* = \hat{\underset{\sim}{\sigma}} + \bar{\underset{\sim}{\rho}}, \tag{2}$$

where $\hat{\underset{\sim}{\sigma}}$ is the linear elastic solution to the problem (i.e. the solution assuming $\dot{v} = 0$) and $\bar{\rho}$ is a time constant residual stress field. The optimal work bound required the condition that the accumulated creep strain per cycle due to $\underset{\sim}{\sigma}^*$ was compatible. It was argued that this stress history corresponded to the *actual* solution when the cycle time was zero, and therefore provided a good approximation when the cycle time was small compared with the material time scale. Although the solution provided a bound on the inelastic work, it also appears that generally the displacements predicted by this solution are greater than those that would be predicted by a complete solution.

In the dynamic problem discussed here, the bounding theory is described in [2], although the completely analogous arguments to these in [1] are not included. In fact the extension to the dynamic situation is quite trivial and merely requirements the reinterpretation of $\hat{\underset{\sim}{\sigma}}$ in (2) as the dynamic elastic solution.

The procedure adopted in this paper is as follows. In Section 2 we derived the exact solution assuming that the cycle time is infinitesimally small, thereby obtaining a solution of the form (2). In Section 3 we derive this solution for two simple problems involving the vibration of a beam. These solutions are interpreted in terms of a reference stress history and thereby directly relate the behaviour of these structures to a beam flexure test composed of a varying sinusoidal moment superimposed upon a content moment. Although the correlation is necessarily approximate the accuracy is remarkably good. We therefore provide an extension to the dynamic situation of the reference stress method which has been used for static loading for some years (a review of this method is given by Marriott [3]). In Section 4 an approximate analysis is described which provides a solution of sufficient accuracy over most of the frequency range.

In the final Section 5, experiments on lead beams are described which indicate that the theoretical predictions may be adequate for all practical purposes.

The methods described in this paper indicate that relatively simple calculations can provide considerable insight into the behaviour of creeping structures under dynamic loading conditions. The extension of these methods to more complex structures will be discussed in a future paper.

2. ANALYTIC PROCEDURE

Consider a structure which is subjected to dead loads P_i over its surface, body forces F_i and dynamic loads $P_i(t)$ and perhaps dynamic displacements $U_i(t)$. The dynamic loads are assumed cyclic with a cyclic time Δt.

The material possesses a stress history $\underset{\sim}{\sigma}$ and strain history $\underset{\sim}{\epsilon}$. These quantities may be interpreted as stress and strain tensors within a continuum or, alternatively, as generalized stresses and generalized strains, such as moments and curvatures, within a beam or plate element. The stresses $\underset{\sim}{\sigma}$ are subject to equilibrium equations which involve accelerations $\ddot{\underset{\sim}{u}}$ arising from displacements $\underset{\sim}{u}$ and density distribution η. Similarly $\underset{\sim}{\epsilon}$ are subject to compatibility equations. Neither of these sets of equations needs to be specified in detail.

The material is described in terms of elastic strains $\underset{\sim}{e}$ and creep strain $\underset{\sim}{v}$ so that,

$$\dot{\underset{\sim}{\epsilon}} = \dot{\underset{\sim}{e}} + \dot{\underset{\sim}{v}} \tag{3a}$$

$$\dot{\underset{\sim}{e}} = \mathbf{C}\dot{\underset{\sim}{\sigma}} \tag{3b}$$

and

$$\dot{\underset{\sim}{v}} = \dot{v}_0 f(\underset{\sim}{\sigma}/\sigma_0) \tag{3c}$$

where C denotes a matrix (or tensor) of elastic constants and f some vector (or tensor) function of $\underset{\sim}{\sigma}$. For convenience we assume f is homogeneous of order n in the components of $\underset{\sim}{\sigma}$, but this assumption is not essential to the arguments.

The elastic solution to the problem, which we denote by $\hat{\underset{\sim}{\sigma}}$, arises when we set $\dot{\underset{\sim}{v}} = 0$. This solution depends upon the cycle time Δt, and, depending upon the initial condition at $t = 0$, will contain some transient response. We will be concerned with times sufficiently large (in paractice a few cycles after $t = 0$) for steady state conditions to have set in, so that the elastic solution is cyclic with period Δt, and all free vibration of the body is ignored. We further assume that the stress history has achieved a cyclic state so that $\underset{\sim}{\sigma}(t) = \underset{\sim}{\sigma}(t + \Delta T)$.

The eqns (3) may be expressed in two alternative forms, in terms of the variation of field quantities within a cycle and in terms of the variation from cycle to cycle.

To evaluate the behaviour within a cycle we transform to time variable τ and t_0 where

$$t = t_0 + \tau \Delta t.$$

Hence t_0 denotes a time at the beginning of a cycle and $0 \leq \tau \leq 1$. Further we introduce a characteristic elastic modulus E (such as the uniaxial Young's modulus) and a time t^* which equals the time for creep strain to equal a characteristic elastic strain (such as uniaxial elastic strain) at stress σ_0

$$t^* = \sigma_0/(E\dot{v}_0).$$

In terms of these quantities eqns (3) becomes

$$\mathring{\underset{\sim}{\epsilon}} = \mathring{\underset{\sim}{e}} + \mathring{\underset{\sim}{v}} \tag{4a}$$

$$\mathring{\underset{\sim}{e}} = C' \mathring{\underset{\sim}{\Sigma}}(\sigma_0/E) \tag{4b}$$

$$\mathring{\underset{\sim}{v}} = (\Delta t/t^*)(\sigma_0/E) f(\underset{\sim}{\Sigma}) \tag{4c}$$

where

$$\underset{\sim}{\Sigma} = \underset{\sim}{\sigma}/\sigma_0, \quad C' = C/E \quad \text{and} \quad \frac{de}{d\tau} = \mathring{e} \text{ etc.}$$

Note that both equilibrium and compatibility equations are independent of t^*.

If σ_0 is chosen so that $\underset{\sim}{\Sigma}$ is of order unity then $\mathring{\underset{\sim}{v}}$ becomes small compared with $\mathring{e}$ if $t^* \gg \Delta t$. In the limit as $\Delta t/t^* \to 0$ then the eqns (4) become identical to the elastic equations and hence

$$\underset{\sim}{\sigma}(\tau, t_0) = \hat{\underset{\sim}{\sigma}}(\tau) + \underset{\sim}{\rho}(t_0) \tag{5}$$

where $\underset{\sim}{\rho}(t_0)$ is a static residual stress field existing at time $t = t_0$, i.e. $\rho(t_0)$ satisfies the static equilibrium equations and is in equilibrium with zero applied loads on S_T.

The accumulation of strain in time t^* is given by

$$t^*(\epsilon(t_0 + \Delta t) - \epsilon(t_0))/\Delta t = t^* \dot{\underset{\sim}{\epsilon}} = t^* \int_0^1 f(\hat{\underset{\sim}{\sigma}}(\tau) + \underset{\sim}{\rho}(t_0))\, d\tau, \tag{6}$$

most remain finite in the limiting process and most satisfy the compatibility conditions.

Although these conditions are only exactly satisfied in the limit as $\Delta t/t^* \to 0$ they should provide a satisfactory approximation when $\Delta t/t^*$ is small. The boundary value problem so posed requires the evaluation of a constant static equilibrium stress field ρ such that the creep strain accumulated over time t^* (or equivalently, over a cycle) should be compatible. Effectively the analysis assumes that the rate of change of stress during a cycle is so rapid that the response of the structure is purely elastic. The creep strain accumulates within each cycle is insufficient to effect these rapid changes, but will, over a period of time, be sufficiently pronounced to form a distribution of residual stresses which allows accumulation of strains which are compatible with a displacement field.

The analysis effectively uncouples the dynamic response of the strucutre from the long term creep deformation and the two aspects of the behaviour are made compatible through the compatibility of the average strain rate $\dot{\bar{\epsilon}}$ of eqn (6). The uniqueness of the residual stress field $\bar{\rho}$ so defined was proven in (1) for quasi-static case for the potential flow law

$$\dot{v} = \frac{\dot{v}_0}{\sigma_0{}^n}\frac{\partial}{\partial \underline{\sigma}}\{\phi^{n+1}(\underline{\sigma})/n+1\} \tag{7}$$

where ϕ denotes a homogeneous function of degree one. The proof remains essentially unchanged for the dynamic case and will not be repeated here.

In the next section the analysis is carried through for a beam problem.

3. VIBRATION OF A BEAM

Consider the beam configuration shown in Fig. 1. A uniform beam of length l, is encastre at $x = 0$ and subjected to prescribed lateral displacement at $x = l$ of amplitude δ and frequency $\omega/2\pi$

$$y(l) = \delta \cos \omega t. \tag{8}$$

The beam has uniform cross section with second moment of area I and area A. The material has Young's modulus E and density γ. A constant load w/unit length acts vertically on the beam.

In the context of this problem, adopting the assumptions of classical linear beam theory, there exists a single generalized stress, the moment M and corresponding generalized strain, the curvature

$$\kappa = \frac{\partial^2 y}{\partial x^2}, \tag{9}$$

where y denotes the lateral deflection of the beam. Corresponding to eqn (3) we write

$$\dot{\kappa} = \dot{\kappa}_e + \dot{\kappa}_v \tag{10a}$$

$$\kappa_e = -M/EI \tag{10b}$$

$$\dot{\kappa}_v = -\dot{\kappa}_0(M/M_0)^n, \tag{10c}$$

where $\dot{\kappa}_0$ denotes the creep curvature rate corresponding to moment M_0.

The elastic response of the beam has two components, those due to the imposed end deflection and those due to the applied load.

The elastic dynamic response of the beam is governed by the differential equation (see for example (4))

$$\frac{\partial^2 y}{\partial t^2} + a^2\frac{\partial^4 y}{\partial x^4} = 0, \quad a^2 = EIg/A\gamma, \tag{11}$$

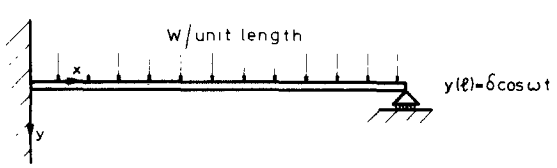

Fig. 1. The beam problem.

and the steady state solution for end deflections given by (8) is,

$$y = (\delta \cos \omega t / P)((\sin kl + \sinh kl)(\cos kx - \cosh kx) - (\cos kl + \cosh kl)(\sin kx - \sinh kx))$$

where

$$k = \sqrt{(\omega / a)}$$

and

$$P = 2(\sinh kl \cos kl - \sin kl \cosh kl).$$

Resonant frequencies correspond to the roots of $P = 0$. The elastic bending moment distribution is given by

$$\begin{aligned} \hat{M} &= -EI\frac{\partial^2 y}{\partial x^2} \\ &= (EIk^2\delta \cos \omega t / P)((\sin kl + \sinh kl)(\cos kx + \cosh kx) \\ &\quad - (\cos kl + \cosh kl)(\sin kx + \sinh kx)). \end{aligned} \tag{12}$$

The complete moment distribution corresponding to $\underset{\sim}{\sigma}^*$ becomes

$$M^* = \hat{M}(x, t) + wx(l - x)/2 + R(l - x) \tag{13}$$

where R donotes the sum of the reactions at $x = l$ due to the elastic response to w and the residual stress field. The value of R becomes determinate from the condition that the average accumulated creep curvature over a cycle

$$\dot{\bar{\kappa}} = -(\omega / 2\pi) \int_0^{2\pi/\omega} (\dot{\kappa}_0 / M_0{}^n) M^{*n} \, dt, \tag{14}$$

shall be compatible with zero displacement at $x = l$. (This condition replaces the compatibility equations of the general argument). By the principle of virtual work, this condition is satisfied provided $\dot{\bar{\kappa}}$ satisfies

$$\int_0^l \dot{\bar{\kappa}}(l - x) \, dx = 0. \tag{15}$$

Combination of eqns (12)–(15) yields a single transcendental equation for R, which may be solved numerically by Newton's method. The average displacement rate may be evaluated by integration of $\dot{\bar{\kappa}}$ to yield $\dot{\bar{y}}$. The most convenient way of evaluating this quantity is from the principle of virtual work corresponding to a point unit load applied at some arbitrary point x' along the beam, with no support at $x = l$:

$$\dot{\bar{y}} = \int_0^{x'} (x' - x) \dot{\bar{\kappa}}(x) \, dx. \tag{16}$$

These calculations were carried out in terms of a number of non-dimensional quantities

$$\left. \begin{aligned} &K = kl = l\sqrt{(\omega / a)}, \quad A = \delta EI / wl^4 \\ &\dot{\bar{y}} = \dot{\bar{y}}(M_0 / wl^2)^n / l^2 \dot{\kappa}_0 \\ &\bar{R} = R / wl, \quad \tau = t / \Delta t = \omega t / 2\pi. \end{aligned} \right\} \tag{17}$$

The maximum elastic deflection under the action of applied loads w is given by

$y_{max} = 0.0054\, wl^3/EI$, and therefore a value of $A = 0.00054$ would correspond to the situation when the amplitude of the end displacement δ equals the maximum static elastic deflection. Variation in K corresponds to variation in frequency ω and the natural frequencies correspond to $K_1 = 3.927$, $K_2 = 7.069$, $K_3 = 10.210$ etc., the roots of $P = 0$. the nondimensional displacement rate $\dot{y}$ therefore depends upon K, A and n. The eqns (10)–(14) may easily be re-expressed in terms of these quantities but will not be re-written here in their non-dimensional form.

The eqn (15) was solved for $\bar{R}$ for a range of values of K and A, and values of $n = 3, 5, 7$ and 9. For $n = 1$ there is no interaction between the dynamic and static loading and the solution would always be the static solution.

First consider the case $A = \delta = 0$, the static problem. The maximum value of $\dot{y} = \dot{y}_{max}$ occurs near $x = 0.6\, l$ for all values of n. For such problems, the reference stress method [3] provides a useful method of reducing the solution to a convenient form. Introducing a parameter α into the expression for $\dot{y}$ we obtain

$$\dot{y} = (\dot{y}\alpha^n) l^2 \dot{\kappa}_0, \tag{18a}$$

where

$$M_0 = wl^2/\alpha. \tag{18b}$$

If it is possible to find a value of α so that $\dot{y}\alpha^n$ is constant and independent of n then (18a) provides an expression for $\dot{y}$ directly in terms of the curvature rate $\dot{\kappa}_0$ corresponding to a reference moment M_0, independent of material constants, thereby directly relating the structural behaviour to a material test. Of course no such α exists, but it is usually possible to find a value of α for which $\dot{y}\alpha^n$ changes only slightly over a range of n values. A value of α was calculated which gave identical values of $\dot{y}_{max}\alpha^n$ for $n = 3$ and $n = 11$, and the results are given in Table 1. Hence we infer that the solution

$$\dot{y}_{max} = 0.05736\, l^2 \dot{\kappa}_0$$

where

$$M_0 = wl^2/12.650, \tag{19}$$

provides an overestimate with a maximum error of about 5% over the range of values of n. This value of M_0 may be compared with the upper bound value given by Ponter and Leckie (5)

$$M_0 = \lambda M_p, \quad w = \lambda w_L, \tag{20}$$

where w_L denotes the plastic limit load corresponding to a plastic bending moment M_p. As

$$w_L = 12\, M_p/l^2, \tag{21}$$

Table 1. Static creep solution ($A = 0$)

Beam with built-in and simply supported ends
$\alpha = 12.650$

n	$\dot{y}\alpha^n$
3	0.05736
5	0.05461
7	0.05506
9	0.05736

Beam with both ends simply supported
$\alpha = 8.482$

n	$\dot{y}\alpha^n$
3	0.09898
5	0.09584
7	0.09682
9	0.09898

then

$$M_0 = wl^2/12,$$

which provides a slightly higher value than that of (19).

If the reference stress method possesses an extension to the dynamic case, the reference stress history may be expected to be of the form

$$M_0(t) = wl^2/\alpha + m_0 \cos \omega t, \tag{22}$$

where m_0 denotes the amplitude of a fluctuating moment history which corresponds to the dynamic response of the structure. Clearly m_0 will depend upon both A and K. As the dynamic stresses are proportional to the activating amplitude δ, it suggests that m_0 may be expressed in the form

$$m_0 = (wl^2/\alpha)AZ, \tag{23}$$

where Z denotes a function of K only. For consistency with (19), the maximum deflection rate would be given by

$$\dot{y}_{max} = 0.05736\, l^2(\omega/2\pi)\int_0^{2\pi/\omega} \dot{\kappa}_0(t)\,\mathrm{d}t, \tag{24}$$

where $\dot{\kappa}_0$ is the curvature rate corresponding to the moment history (22). To test whether this conjective has any validity values of Z were calculated which would make these expressions exact for various values of n, A and K. Table 2 shows the results for this calculation for two values of K. Values of Z were evaluated by Newton's method, using the values of $\dot{y}$ computed from the structural calculation. The range of values of A covers several orders of magnitude in the value of $\dot{y}$. The values of AZ vary from 3.10^{-3} to 3 and correspond to the situation from where the dead loads dominate to where the dynamic loads dominate. For both values of K the variation is remarkably small and the maximum differences are of the order of 4% for $K = 2.5$ and 2% for $K = 3.9$. The latter value of K lies close to the resonant frequency ($K_1 = 3.927$) and

Table 2. Values of Z for beam problem in Fig. 1

		$K = 2.5$		
A	$n = 3$	$n = 5$	$n = 7$	$n = 9$
0.0001	31.636	30.889	30.638	30.534
0.0002	31.636	30.888	30.636	30.531
0.0003	31.636	30.887	30.634	30.526
0.0004	31.635	30.886	30.631	30.520
0.0005	31.635	30.885	30.626	30.511
0.0006	31.635	30.883	30.621	30.501
0.0007	31.635	30.880	30.615	30.489
0.0008	31.634	30.878	30.607	30.475
0.0009	31.634	30.875	30.599	30.459
0.001	31.634	30.871	30.590	30.441
		$K = 3.9$		
A	$n = 3$	$n = 5$	$n = 7$	$n = 9$
0.0001	3506.9	3494.6	3485.6	3479.1
0.0002	3504.6	3487.2	3478.6	3474.2
0.0003	3499.4	3485.0	3477.5	3473.7
0.0004	3492.7	3483.3	3477.1	3473.7
0.0005	3486.3	3489.4	3475.7	3473.4
0.0006	3480.8	3473.2	3472.3	3471.8
0.0007	3476.5	3465.7	3466.4	3468.0
0.0008	3473.1	3457.8	3458.4	3461.6
0.0009	3470.4	3450.1	3449.2	3453.0
0.001	3468.3	3443.0	3439.6	3442.6

the values of Z are correspondingly higher. For all practical purposes we may take Z as the higher value, thereby achieving an over-estimate of the maximum deflection rate. Hence we arrive at the approximate relationship

$$\dot{y}_{max} = 0.05736\, l^2(\omega/2\pi) \int_0^{2\pi/\omega} \dot{\kappa}_0(t)\, dt$$

$$M_0(t) = (wl^2/12.650)(1 + (\delta EI/wl^4)Z \cos \omega t), \qquad (25)$$

where $Z = 31.6$ for $K = 2.5$ and $Z = 3500$ for $K = 3.9$. These values are overestimates and the exact values obtained by these processers will obviously vary with the method adopted. The principal conclusion is that such a correlation is possible and suffers from a relatively small error.

To find the variation of Z with K requires the evaluation of Z for a fixed value of n and A. As, in both the cases of Table 2, small values of both A and n have produced the largest values of Z, a sequence of such calculations were carried out for $n = 3$ and $A = 0.001$ and the results are displayed in Fig. 2.

Although the position of the maximum deflection remained relatively constant in position, the shape of the deflection rate contours do change with frequency and a sequence of such contours are shown in Fig. 3.

The method was also applied to the statically determinate problem which arises when the beam is assumed to be simply supported at $x = 0$. In this case the dynamic elastic solution is given by

$$y = \delta \cos \omega t (\sin kl \sinh kx + \sinh kx \sin kx)/P$$

$$P = 2 \sin kl \sinh kl. \qquad (26)$$

The resonant frequencies are given by $K = 2m\pi$, $m = 1, 2, 3$ etc. The history M^* becomes

$$M^* = \hat{M} + wx(l - x)/2, \qquad (27)$$

as there exists no residual stress field. The displacement rate at a point along the beam is given by

$$\dot{y}(x') = \int_0^{x'} \bar{M}\dot{\kappa}\, dx, \qquad (28)$$

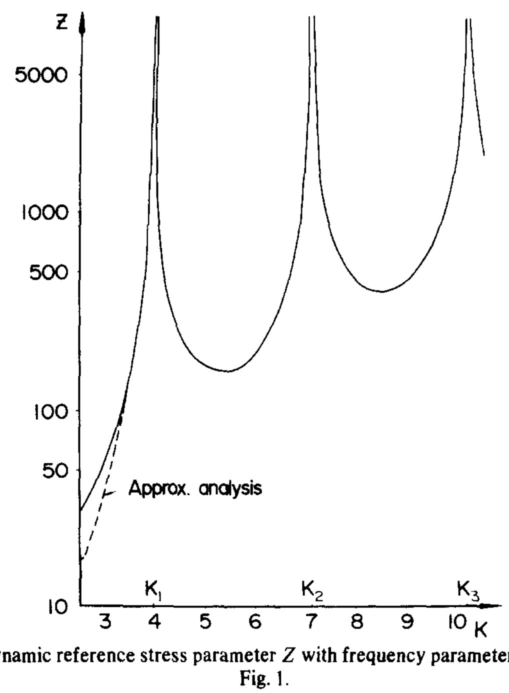

Fig. 2. Variation of dynamic reference stress parameter Z with frequency parameter K for beam problem of Fig. 1.

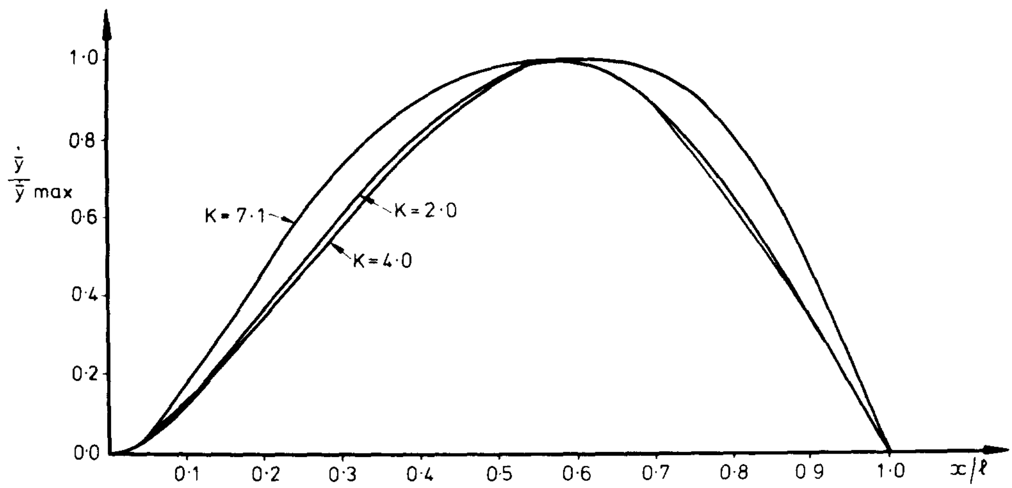

Fig. 3. Beam problem of Fig. 1. Creep deflections for $K = 2.0, 4.0$ and 7.1.

where

$$\bar{M} = -x'(l-x), \quad x' \leq x$$
$$= -x(l-x'), \quad x \geq x'. \tag{29}$$

The moment distribution $\bar{M}$ corresponds to the moments at x due to a point load at x'. The curvature rate $\dot{\kappa}$ is given by (14) and (27).

The procedure adopted was identical to that of the previous example and the appropriate value of α for $A = 0$ and associated values of $\dot{y}\alpha^n$ are given in Table 1. Equation (21) yields the value of $\alpha = 8$ giving a slightly higher value of M_0, as expected. The values of Z for two values of K, ($K_1 = 3.1416$) are given in Table 3 which shows only slight variations with n and Z although, in this case, the largest values correspond to larger values of n and A. Hence corresponding to eqn (25) we obtain

$$\dot{y}_{max} = 0.09898\, l^2\omega/2\pi \int_0^{2\pi/\omega} \dot{\kappa}_0\, dt$$

$$M_0(t) = (wl^2/8.482)(1 + (\delta EIZ/wl^4)\cos \omega t),$$

where $Z = 9.66$ for $K = 2.0$ and $Z = 9.09$ for $K = 3.1$.

Table 3. Value of Z for simply supported beam

		$K = 2.0$		
A	$n = 3$	$n = 5$	$n = 7$	$n = 9$
0.001	9.6229	9.6227	9.6225	9.6224
0.003	9.6230	9.6227	9.6225	9.6826
0.001	9.6230	9.6227	9.6225	9.6224
0.003	9.6230	9.6227	9.6226	9.6226
0.01	9.6232	9.6233	9.6237	9.6242
0.03	9.6243	9.6264	9.6293	9.62327
0.1	9.6269	9.6417	9.6534	9.6618
		$K = 3.1$		
A	$n = 3$	$n = 5$	$n = 7$	$n = 9$
0.0001	901.20	905.04	906.31	908.81
0.003	901.20	905.05	907.33	908.85
0.001	901.21	905.14	907.46	908.98
0.003	901.21	905.27	907.60	909.11
0.01	901.21	905.31	907.66	909.18
0.03	901.21	905.31	907.66	909.19
0.1	901.21	905.31	907.67	909.19

4. APPROXIMATE PERTURBATION ANALYSIS

Although the method of analysis described in the last section can be carried out without difficulty for a simple beam problem, for more complex problems the analysis becomes tedious. As Z remains relatively constant with A, the non-dimensional dynamic amplitude, it may be expected that a good approximate value can be found by considering the extreme case when A is large or equivalently when w is small compared with the maximum value of $\hat{M}/l^2$. Such an analysis can be achieved by expanding the governing equation for the beam problem, eqns (15) and (16) in a series in w and terminating the series at the linear term in w. The first order solution so obtained may then be correlated with the corresponding expansion of the reverence stress solution eqns (23) and (24) yielding Z for a given value of K. The details of this analysis are given in the Appendix, and the resulting values of Z for $N = 3$ are shown as the dashed line in Fig. 2. The values are indistinguishable from the exact analysis except for small values of K.

This approximate analysis can be shown to be equivalent to solving a linear problem, and yields a solution directly by integration.

5. EXPERIMENTAL STUDY

A sequence of experiments were conducted on lead beams at room temperature to assess the accuracy of these theoretical predictions. Beams of length 30.98 cm, width 2.54 cm and depth 0.635 cm were cut from a sheet of commercially pure lead which had been cast into an ingot 1.27 cm thick and then reduced by rolling in two directions.

Measurements of stationary state creep rate were made for two test specimens which were machined from the sheet in a direction parallel to the axis of the beams, thereby minimizing any effects of anisotropy which may be present. Each test specimen contained three separate lengths of constant cross-sectional area, to which strain/gauges were attached. Hence, by suitable choice of loads, six creep rates were obtained within the stress range 0.4 to 0.8 kg/mm^2. A graph of log σ against log $\dot{v}^s$ where $\dot{v}^s$ is the stationary state creep rate, is shown in Fig. 4, and it can be seen that the data correlates well with $n = 9$.

The beams were mounted with one end restrained from vertical movement and rotation in a clamp. The other end was mounted in a ball race pivot which allowed rotation and horizontal movement. The ball race pivot was attached to the actuator of a Losenhausen servo-hydraulic

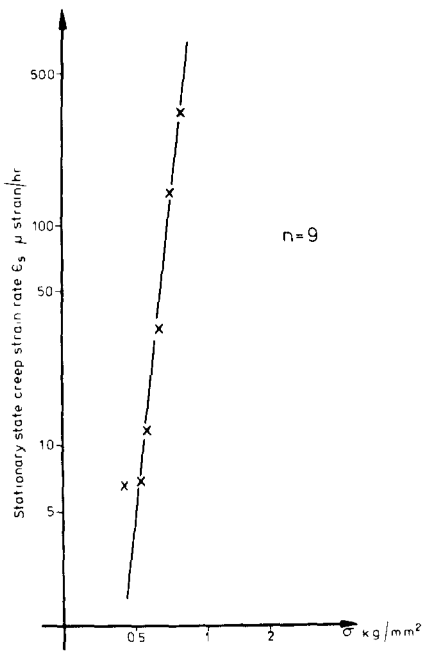

Fig. 4.

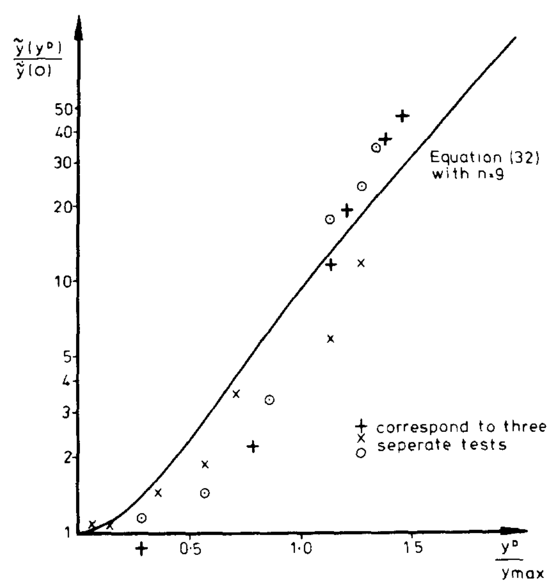

Fig. 5.

Fig. 4. Stationary state creep rates of lead.

Fig. 5. Experimental values of $\tilde{y}(y^D)$, the average maximum deflection rate for increasing values of dynamic elastic deflection y^D. y_{max} is the maximum static elastic deflection.

testing machine which could be programed to move vertically at a preset amplitude and frequency. The central deflection was measured by a thin beam extensometer. This consisted of a thin steel cantilever which had strain gauges attached to its surface near the clamped end, and provided both an accurate measurement of displacement and produced a negligible restraint on the vibration of the beam. The response of the extensometer was recorded on a brush high speed dynamic pen recorder.

The principal difficulty of the experimental configuration was that the Losenhausen was capable of producing adequate amplitudes of beam end deflection only at frequencies which were small compared with the lowest natural frequency of the lead beam which was approximately 50 Hz. The experiments were therefore conducted at frequencies very close to the lowest natural frequencie, so that substantial vibration of the beam could be achieved with small beam end amplitude δ. In the vicinity of the natural frequency it was found from computed values that

$$\delta EIZ/wl^4 = 0.50\, y^D/y_{max}, \tag{31}$$

where y^D denotes the amplitude of the elastic vibration at the mid-point of the beam and $y_{max} = 0.0054\, wl^3/EI$, the maximum elastic static deflection under self-weight. Taking the density of lead as $11300\,\text{kg/m}^2$ and $E = 1.4 \cdot 10^3\,\text{kg/mm}^2$ we find $y_{max} = 0.139\,\text{mm}$.

Hence from eqn (25) the theory predicts that the maximum average deflection rate $\dot{y}(y^D)$ as a function of the dynamic amplitude y^D is given by

$$\dot{y}(y^D)/\dot{y}(0)) = \int_0^{2\pi} (1 + 0.50\,(y^D/y_{max}) \cos \tau)^n \, d\tau, \tag{32}$$

and $n = 9$.

Tests were conducted on three nominally identical beams. Each beam was first allowed to creep under self-weight until a stationary creep rate was achieved. A small end amplitude δ was then commenced and the response of the beam at the position of maximum deflection rate was recorded for a period until the average creep rate $\dot{y}$ achieved a constant value. These time intervals were of the order of 30 min. From the chart traces both y^D and $\dot{y}$ could be calculated. The end amplitude was then increased, and by this means a set of values of $\dot{y}$ were found for increasing y^D. Maintenance of a constant y^D over an extended time period proved difficult but could be generally maintain within 10% of a constant value. The experimental points so obtained for three separate beam tests are shown in Fig. 5, where a fair degree of scatter can be seen. The prediction of the formula (32) is shown and can be seen to overestimate $\dot{y}$ at lower amplitudes. As the reference stresses are somewhat less than those which yield a value of $n = 9$, this may be explained by the fact that n usually reduces with stress.

Considering the simplicity of the material model and the inherent difficulties of carrying out controlled experiments of this type, the correlation can be considered satisfactory. As the value $n = 9$ is high, the vibration of the beam causes a rapid increase in the creep rate. A value of y^D equal to the static elastic deflection causes an order of magnitude increase in the creep rate. Naturally, smaller values of n will produce a less dramatic effect. Nevertheless it is clear that the presence of vibration can have a considerable effect on the creep rate of structures.

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APPENDIX: APPROXIMATE SOLUTION OF BEAM PROBLEMS

The analysis assumes that w is small and eqn (14) is expanded as a series in w:

$$\frac{\Delta\kappa}{\Delta t}=\frac{\omega}{2\pi}\frac{\overset{\circ}{\kappa}}{M_0{}^n}\int_0^{2\pi/\omega}\left\{\hat{M}+R(0)(l-x)^n+nw\left[\frac{x(l-x)}{2}+R'(0)(l-x)\right][\hat{M}+R(0)(l-x)]^{n-1}\right\}\,\mathrm{d}t+0(w^2) \tag{A1}$$

where $R=R(w)$ and $R'=(\mathrm{d}R/\mathrm{d}w)$.

The first term corresponds to the curvature rate when $w=0$. The corresponding end reaction $R(0)=0$ and hence the contribution from the first term is zero. Ignoring terms involving w^2 the compatibility eqn (15), applied to the second term of (A1) yields

$$R'(0)=\frac{-\int_0^l\left(\frac{x(l-x)}{2}\right)F^{n-1}(l-x)\,\mathrm{d}x}{\int_0^l(l-x)^2F^{n-1}\,\mathrm{d}x}, \tag{A2}$$

where $F=(\sin kl+\sinh kl)(\cos kx+\cosh kx)-(\cos kl+\cosh kl)(\sin kx+\sinh kx)$. The deflection rate may now be evaluated by substituting (A2) and (A1) into (16), which was evaluated at $x=0.57\,l$.

The reference stress expression (24) may be expanded in a similar way to yield

$$\frac{\Delta y}{\Delta t}=0.05736\,l^2\frac{\dot{\kappa}^0}{M_0{}^n}\left(\frac{l^2}{12.650}\right)n\left(\frac{\delta EIZ}{l^2\cdot 12.650}\right)^{n-1}\int_0^{2\pi}\cos^{n-1}\tau\,\mathrm{d}\tau+0(\omega^2).$$

Equating this expression to that given by (16) yields Z directly.